The Number of the Two Dimensional Run Length Constrained Arrays

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Abstract. First, a new framework describing the transfer matrices for the two dimensional run length constrained arrays (codes) is introduced, and some important properties of the transfer matrices are derived in this framework. Then, using these properties, it is shown that the numbers \( N(m,n) \) of the two dimensional binary arrays satisfying the \((1,\infty)\) - run length constraint is expressible by a linear recurrence equation of a fixed order, approximately equal to \( \dim(T_m) / 2 \). Finally, to demonstrate the effectiveness of the result obtained, some numerical examples are presented.

Keywords: Run length constrained array, transfer matrix, information capacity; data storage

1. Introduction

Run length constrained arrays (codes) are widely used in digital data recording and transmission. Its generalization to the two-dimensional case is of potential interest in the page-oriented information storage technologies, such as holographic storage. A one-dimensional binary sequence is said to satisfy a \((d,k)\)-run length constraint if every run of zeros in the sequence has length at least \( d \) and at most \( k \). Similarly an 2-dimensional binary array is said to satisfy a \((d,k)\)-run length constraint if it satisfies the one-dimensional - run length constraint both horizontally and vertically. Let the number of 2-dimensional binary arrays of size \( m \times n \) satisfying the \((d,k)\)-run length constraint be denoted by \( N_{d,k}(m,n) \). Then the information capacity is defined as

\[
C_{d,k} = \lim_{m,n \to \infty} \frac{1}{mn} \log_2 N_{d,k}(m,n)
\]

which may be interpreted as the maximum number of bits of information that can be stored asymptotically per unit volume. Recently a great deal of effort has been made for evaluating the numbers \( N_{d,k}(m,n) \) and the limit \( C_{d,k} \), in particular, for investigating the existence of the limit and lower and upper bounds on \( C_{d,k} \) for various values of \((d,k)\). See, e.g., [1]-[6] and the references herein.

This paper focuses on a special case of the 2-dimensional constrained arrays with \( d = 1 \) and \( k = \infty \). For this case, the number \( d_m := N_{1,\infty}(m,1) \) is well known as the Fibonacci number. Further it is easily seen by exchanging the roles of 0 and 1 that \( C_{1,\infty} = C_{0,1} \). For notational simplicity, hereafter \( N_{1,\infty}(m,n) = N_{0,1}(m,n) \) will be denoted simply by \( N(m,n) \).

This special case was extensively studied in [1]-[4] and also in [5], and many important results have been obtained. One of important and interesting results among them is the work by Calkin and Wilf [1]. In this work, they used the fact that the number \( N(m,n) \) can be expressed in the form

\[
N(m,n) = 1' T_m^{n-1} 1, \quad m,n \geq 1
\]

where \( 1' \) indicates the column vector with all entries equal to 1 and a compatible dimension, \( 1' \) denotes the transpose, \( T_m \) is the transfer matrix of the two-dimensional \((1,\infty)\) - run length constraint problem introduced
in [1]. The transfer matrix $T_m$ is a symmetric $d_m \times d_m$ matrix with all entries equal to 0 or 1 and the order $d_m$ is computed in the following recursive manner:

$$
\begin{cases}
    d_1 = d_0 = 1 \\
    d_m = d_{m-1} + d_{m-2}, & m \geq 1.
\end{cases}
$$

Therefore all the properties of $N(m,n)$ can be obtained through the corresponding transfer matrix $T_m$.

$N(m,n)$ even for relatively large values of $n$ and $m$. Finally to demonstrate effectiveness of our results, the recursive equations of $N(m,n)$ for $m = 2, 3, 4$ will be computed as numerical examples.

### 2. Description of Transfer Matrices

In this section, we first introduce a new framework for representing the transfer matrix $mT$ defined in Calkin and Wilf [1], and investigate their important properties. First let $\{0,1\}^{p \times q}$ denote the set of all $p \times q$ - dimensional matrices with all entries in $\{0,1\}$. Then, we have the following results.

**LEMMA 1.** Let us introduce the two sequences of matrices $V(m)$ and $W(m)$ for $m \geq 1$ by the following recursive formula:

$$
\begin{align*}
V^{(1)} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & V^{(m)} &= \begin{bmatrix} V^{(m-1)} \end{bmatrix}^{d_m-1} 1, & m \geq 2, \\
W^{(1)} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & W^{(m)} &= \begin{bmatrix} W^{(m-1)} \end{bmatrix}^{d_m-1} 1, & m \geq 2
\end{align*}
$$

where the notation $X^{[k]}$ indicates the sub-matrix composed of the first $k$ rows of a matrix $X$, and $0$ is a zero column vector with a compatible dimension. Further let us represent the matrices $V^{(m)}$, $W^{(m)}$ as the row vectors:

$$
V^{(m)} = \begin{bmatrix} v_1^{(m)} \\ \vdots \\ v_{d_m}^{(m)} \end{bmatrix} \in \{0,1\}^{d_m \times m}, & W^{(m)} = \begin{bmatrix} w_1^{(m)} \\ \vdots \\ w_{d_m}^{(m)} \end{bmatrix} \in \{0,1\}^{d_m \times m}
$$

Then the following statements hold.

(i) The row vectors $v_1^{(m)}, \ldots, v_{d_m}^{(m)}$ are all distinct and so are $w_1^{(m)}, \ldots, w_{d_m}^{(m)}$.

(ii) For each $i = 1, \ldots, d_m$, $v_i^{(m)}$ and $w_i^{(m)}$ coincide in the reversed order, that is, if $v_1^{(m)} \equiv \cdots \equiv v_i^{(m)} \equiv \cdots \equiv v_{d_m}^{(m)}$ then $w_1^{(m)} \equiv \cdots \equiv w_i^{(m)} \equiv \cdots \equiv w_{d_m}^{(m)}$.

(iii) For each $i = 1, \ldots, d_m$, both $v_i^{(m)}$ and $w_i^{(m)}$ have no two consecutive 1’s in their components, respectively.

(iv) Set $\{v_1^{(m)}, \ldots, v_{d_m}^{(m)}\}$ coincides with the set of all possible row vectors in $v \in \{0,1\}^{d_m \times m}$ having no two consecutive 1’s.

Next, let $p, q \geq 1$ be arbitrary integers, and for any two matrices

$$
V = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} \in \{0,1\}^{p \times q}, & W = \begin{bmatrix} w_1 \\ \vdots \\ w_p \end{bmatrix} \in \{0,1\}^{p \times q},
$$

define the two operations $\wedge$ and $\lor$ by

$$
\begin{align*}
V \wedge W &= \begin{bmatrix} v_1 \wedge w_j \end{bmatrix}^{p \times q} \in \{0,1\}^{p \times q} \\
V \lor W &= \begin{bmatrix} v_1 \lor w_j \end{bmatrix}^{p \times q} \in \{0,1\}^{p \times q}
\end{align*}
\quad \text{where } v_1 \wedge w_j := \begin{cases} 1 & \text{if } \langle v_1, w_j \rangle = 0 \\
0 & \text{otherwise.}
\end{cases}
$$

Then the following theorem can be proved. The statement (ii) has been shown in a different framework in [6], but it would be worthwhile to reprove it because our framework gives a simpler and more straightforward proof.

**THEOREM 1.** The following facts hold.

(i) The transfer matrix $T_m$ is expressed as $T_m = V^{(m)} \lor W^{(m)} = W^{(m)} \lor V^{(m)}$.
(ii) Further \( T_m \) is computed by the following recursive formula:

\[
T_m = \begin{bmatrix} T_{m-1} & (T_{m-1})^{[d_{m-2}]} \end{bmatrix}^{T}, \quad T_j = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} .
\]  

**Proof.** The statement (i) can be verified simply by observing that Lemma 1 ensures that the vectors \( v_1^{(m)}, \ldots, v_{d_m}^{(m)} \) and \( w_1^{(m)}, \ldots, w_{d_m}^{(m)} \) satisfy the conditions for the transfer matrix \( T_{m-1} \) defined in [1].

To prove (ii), letting \( k \geq 2 \) and assuming that (11) is true for \( m = k - 1 \), it will be proved that (11) is also true for \( m = k \). This is done by directly evaluating \( V^{(k)} \). \( \square \)

**LEMMA 2.** Consider two matrices \( V, W \) and a permutation matrix \( P \) represented as

\[
V = \begin{bmatrix} v_1 \\ \vdots \\ v_q \end{bmatrix} \in \{0,1\}^{q \times r}, \quad W = \begin{bmatrix} w_1 \\ \vdots \\ w_q \end{bmatrix} \in \{0,1\}^{q \times r} 
\]

\[
P = [p_1 \cdots p_q] \in \{0,1\}^{q \times q}
\]

where \( v_j, w_j \) are row vectors and each \( p_j \) is a column unit vector. Then the following equality holds true:

\[
(VP) \lor (PW) = (V \lor W)P .
\]  

**Proof:** First note that \( P \) is represented as a permutation of the identity matrix \( I = [e_1 \cdots e_q] \) where \( e_k \) is the column unit vector with \( k \)-th entry equal to 1. Therefore it suffices to verify (6) for a special case such that \( P \) is a permutation matrix of the form

\[
P = [e_1 \cdots e_q]_{i \leftrightarrow j}, \quad 1 \leq i \leq j \leq q 
\]

where \([\cdot \cdot]_{i \leftrightarrow j}\) indicates an interchange of the \( i \)-th and \( j \)-th column vectors. For this permutation matrix \( P \), we have

\[
(VP) \lor (PW) = V_{i \leftrightarrow j} \lor W_{i \leftrightarrow j} = \begin{bmatrix} v_1 \\ \vdots \\ v_q \end{bmatrix}_{i \leftrightarrow j} \lor \begin{bmatrix} w_1 \\ \vdots \\ w_q \end{bmatrix}_{i \leftrightarrow j} = \begin{bmatrix} v_1 \\ \vdots \\ v_q \end{bmatrix} = P(V \lor W)P 
\]

where similarly \([\cdot \cdot]_{i \leftrightarrow j}\) indicates the interchange of the \( i \)-th and \( j \)-th column vectors. \( \square \)

**THEOREM 2.** Let \( V^{(m)}, W^{(m)} \) be given by (4) and (5), and define

\[
P^{(m)} := V^{(m)} \land W^{(m)} \in \{0,1\}^{d_m \times d_m}, \quad m \geq 1 .
\]  

Then, for any \( m \geq 1 \) the following statements hold.

\( P_m \) is an \( d_m \times d_m \) symmetric permutation matrix, and satisfies \( V^{(m)} = P_m W^{(m)} \). Further

\[
T_m^n P_m = T_m^n \forall m, \quad n = 0, \pm 1, \pm 2, \ldots .
\]

3. **Recursive Formulas**

In this section, it is shown that the numbers \( N(m,n) (m,n \geq 1) \) of the two-dimensional binary arrays satisfying the \((1, \infty)\) - run length constraint can be represented by a linear recurrence equation of a fixed order. In particular, it is shown that the order of the linear recurrence equation is approximately equal to \( \dim(T_m)/2 \) for large values of \( m \).

**LEMMA 3.** Let \( P_m \) be given by (14). Then, its trace is given as

\[
\text{tr}(P_m) = \begin{cases} 
\frac{d_m}{2} & \text{if } m \text{ is even} \\
\frac{d_m-1}{2} & \text{if } m \text{ is odd} 
\end{cases}
\]

**Proof.** It follows from (14) that \( \text{tr}(P_m) = \sum_{i=1}^{d_m} (v_i^{(m)} \land w_i^{(m)}) \)
First note that, by virtue of the definition of operation \( \wedge \), \( v_i^{(m)} \wedge w_i^{(m)} = 1 \) if and only if \( v_i^{(m)} = w_i^{(m)} \). In addition to this, since by (ii) of Lemma 1 \( v_i^{(m)} \) and \( w_i^{(m)} \) coincide in the reversed order, therefore if \( v_i^{(m)} \wedge w_i^{(m)} = 1 \) then \( v_i^{(m)} \) must be of the form

\[
v_i^{(m)} = \begin{bmatrix} v_{i,1}^{(m)} & \cdots & v_{i,m/2-1}^{(m)} & v_{i,m/2}^{(m)} & v_{i,m/2+1}^{(m)} & \cdots & v_{i,1}^{(m)} \end{bmatrix}
\]

when \( m \) is even, and

\[
v_i^{(m)} = \begin{bmatrix} v_{i,1}^{(m)} & \cdots & v_{i,(m-1)/2}^{(m)} & v_{i,(m-1)/2+1}^{(m)} & \cdots & v_{i,1}^{(m)} \end{bmatrix}
\]

when \( m \) is odd.

Furthermore, since \( v_i^{(m)} \) has no two consecutive 1's by Lemma 1, it is easily seen from (20) and (21) that a necessary and sufficient condition for \( v_i^{(m)} \wedge w_i^{(m)} = 1 \) is that \( v_i^{(m)} \) is of the form

\[
v_i^{(m)} = \begin{bmatrix} 0 & 0 & v_{i,2}^{(m)} & \cdots & v_{i,1}^{(m)} \end{bmatrix}
\]

if \( m \) is even, and

\[
v_i^{(m)} = \begin{bmatrix} v_{i,1}^{(m)} & v_{i,2}^{(m)} & \cdots & 0 & 0 \end{bmatrix}
\]

if \( m \) is odd.

Now (9) is evaluated. First, for the case of \( m \) being even, the summation is equal to the number of all possible vectors that has the form of (11), which in turn, the form of \( \begin{bmatrix} v_{i,1}^{(m)} & \cdots & v_{i,(m-1)/2}^{(m)} \end{bmatrix} \), and hence the summation is equal to \( d_{(m-2)/2} \). Similarly, for the case of \( m \) being odd, the summation (19) is equal to the number of all possible vectors that has the form of (12), which in turn, the form of \( \begin{bmatrix} v_{i,1}^{(m)} & \cdots & v_{i,(m+1)/2}^{(m)} \end{bmatrix} \), and hence the summation is equal to \( d_{(m-1)/2} \). This completes the proof.

Next, let \( m \geq 1 \) be arbitrarily fixed, and define a sequence of \( d_m \)-dimensional column vectors by

\[
\Phi_k := \begin{bmatrix} \varphi_{k1} \\ \varphi_{k2} \\ \vdots \\ \varphi_{kd_m} \end{bmatrix} := T_m^{k-1}1, \quad k \geq 1,
\]

and a sequence of matrices

\[
\Phi_k = \begin{bmatrix} \varphi_{k1} \\ \vdots \\ \varphi_{kd_m} \end{bmatrix} := [\varphi_1 \varphi_2 \cdots \varphi_k] = \begin{bmatrix} 1 & T_m & \cdots & T_m^{k-1} \end{bmatrix}, \quad k \geq 1.
\]

LEMMA 4. The following statements hold:

(i) The vectors \( \varphi_k \) defined in (13) satisfy \( P_m \varphi_k = \varphi_k, \quad k \geq 1 \)

(ii) The matrices \( \Phi_k \) defined in (14) satisfy \( \text{rank} \Phi_k = \text{rank} \Phi_{k+1} \Rightarrow \text{rank} \Phi_k = \text{rank} \Phi_{k+\alpha}, \quad \forall \alpha \geq 2 \).

**Proof.** The statement (i) can be easily proved using Theorem 2 (ii). In fact, for any \( k \geq 1 \)

\[
P_m \varphi_k = P_m T_m^{k-1}1 = T_m^{k-1}P_m 1 = T_m^{k-1}1 = \varphi_k.
\]

Next to prove (ii), assume \( \text{rank} \Phi_k = \text{rank} \Phi_{k+1} \). Then, first note that \( \text{rank} \Phi_k = \text{rank} \Phi_{k+1} \) implies that \( \varphi_k \) can be expressed as a linear combination of the vectors in \( \Phi_k \), i.e., \( \varphi_k = \Phi_k \zeta \) for some vector \( \zeta \in \mathbb{R}^k \). This fact leads to

\[
\Phi_{k+1} = T_m \Phi_k = T_m \Phi_k \zeta = T_m [\varphi_0 \varphi_1 \cdots \varphi_{k-1}] \zeta = [\varphi_1 \cdots \varphi_{k-1} \Phi_k \zeta \zeta] = [\varphi_1 \cdots \varphi_{k-1} \zeta \zeta] = [\varphi_1 \cdots \varphi_{k-1} \zeta \zeta].
\]

Therefore \( \Phi_{k+1} \) is also a linear combination of \( \varphi_0, \varphi_1, \cdots, \varphi_{k-1} \), and hence \( \text{rank} \Phi_k = \text{rank} \Phi_{k+1} = \text{rank} \Phi_{k+\alpha} \).

Repeating this process, one can verify \( \text{rank} \Phi_k = \text{rank} \Phi_{k+\alpha} \) for all \( \alpha \geq 2 \).

LEMMA 5. Let \( \Phi_k \) be the matrices defined by (14). Then, \( \max \text{rank} \Phi_k \leq \frac{1}{2} [d_m + \text{tr}(P_m)] =: r_m \)

**Proof.** First, recall that \( P_m \) is a symmetric permutation matrix and \( \text{tr}(P_m) \) is given by (18). It is not difficult to see from LEMMA 4 (i) that, for every \( k \geq 1 \), \( \text{tr}(P_m) \) entries in the vector \( \varphi_k \) are fixed by this permutation \( P_m \) and the other entries in \( \varphi_k \) are pair-wisely permuted but the two entries in each pair are equal. That is to say, there exist two subsets \( \{a_1, \cdots, a_q\} \) and \( \{b_1, \cdots, b_p, \gamma_1, \cdots, \gamma_p\} \) of \( \{1, \cdots, d_m\} \) with \( q = \text{tr}(P_m) \) and \( p = (d_m - q)/2 \), independent of \( k \geq 1 \), such that the entries \( \varphi_{a_1}, \cdots, \varphi_{a_q} \) of \( \varphi_k \) are not permuted by \( P_m \) and
the other entries are pair-wisely equal, i.e., \( \phi_k \beta_k = \phi_{k_1} \beta_{k_1}, \cdots, \phi_k \beta_p = \phi_{k_7} \beta_{k_7} \). Therefore, the row vectors of \( \Phi_k \) given in (14) satisfy

\[
\phi_k (\beta_k) = \phi_k (\gamma_1), \cdots, \phi_k (\beta_p) = \phi_k (\gamma_p), \quad \forall k \geq 0.
\]

Therefore there are at least \( p \) linearly dependent vectors in \( \{ \phi_k (1), \cdots, \phi_k (d_m) \} \), and hence

\[
\max_{k \geq 0} \text{rank} \Phi_k \leq d_m - \frac{1}{2} \{ d_m - \text{tr}(P_m) \} = \frac{1}{2} \{ d_m + \text{tr}(P_m) \},
\]

which proves the desired result.

Now it is ready to state and prove our main theorem as follows.

**THEOREM 3.** Let \( m \geq 1 \) be arbitrarily fixed. Consider the numbers \( N(m, n) \) given by (2) and the vectors \( \phi_n \) given by (14), that is,

\[
\begin{aligned}
N(m, n) &= 1' T_{m}^{n-1} 1, \quad m, n \geq 1 \\
\phi_n &= T_{m}^{n-1} 1, \quad n \geq 1.
\end{aligned}
\tag{15}
\]

Then the following statements hold:

(i) The sequence \( \{ \phi_n \} \) satisfies a linear recurrence equation of order \( r_m \), that is,

\[
\phi_n = \beta_1 \phi_{n-1} + \beta_2 \phi_{n-2} + \cdots + \beta_{r_m} \phi_{n-r_m}, \quad n > r_m
\]

where \( \beta_k (k = 1, \cdots, r_m) \) are some constants, independent of \( n \).

(ii) The numbers \( N(m, n) = 1' T_{m}^{n-1} 1 \) \( (n \geq 1) \) also satisfy the same linear recurrence equation as (28), that is,

\[
N(m, n) = \beta_1 N(m, n-1) + \beta_2 N(m, n-2) + \cdots + \beta_{r_m} N(m, n-r_m), \quad n > r_m
\tag{16}
\]

**Proof.** This theorem is easily proved. In fact, by virtue of LEMMA 4, \( \phi_n \) with \( n \geq r_m \) is expressed as a linear recursive equation of the form (28), which can be also written as

\[
T_{m}^{n-1} 1 = \beta_1 T_{m}^{n-2} 1 + \beta_2 T_{m}^{n-3} 1 + \cdots + \beta_{r_m} T_{m}^{n-r_m} 1, \quad n > r_m
\tag{17}
\]

and since \( N(m, k) = 1' T_{m}^{k-1} 1 \) \( (k \geq 1) \), the linear recurrence equation (16) directly follows form (30).

### 4. References


