Variable Structure Control Continuous Traffic flow in Automatic Highway System

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Abstract. In order to get stable and orderly road traffic flow, the speed controller was designed based on the variable structure control theory. The density of each section of road was the control objective, firstly, the target speed got from the first step Lyapunov function, then, control item got from the second step Lyapunov function. The simulation results show that the density stabilized within a certain range, thus avoided road congestion.

Keywords: traffic flow; Lyapunov function; variable structure control; speed controller

1. Introduction

Internal and external factors cause fluctuation of traffic flow, and unstable traffic flow may cause traffic congestion and lower efficiency of the system. Automatic highway system[1-7] provides appropriate feedback control command for vehicles on road based on road traffic condition. Because computer control instead of subjective control, traffic flow can be ordered and stabilized. According to the difference of control objects, there are two main control types including micro control [1-3]and macro control [4-7], and the macro control is the main research area.


This paper presents a new control method for continuous macro traffic flow, which derived from variable structure control theory. The aim is getting stable traffic flow, and the speed controller is designed, under the control the density approaches to specified value, then avoided traffic congestion.

The remainder of this paper is organized as follows: traffic model is given in Section2. Section 3 introduces the variable structure control, and based on Lyapunov function, the control item is deduced in Section 4. In Section 5, numeric simulation proves the effectiveness of our method.

2. Macro Traffic Model

Traffic models include into macro models and micro models, the study target in this paper is the macro traffic model, which proposed by Pageorgiou M [8,9]. In Fig 1, the road divided into many sections, and all vehicles in the same section have the same speed and acceleration rule.

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For the $i$th section, let $v_i$, $\rho_i$, $q_i$, $L_i$ denote speed, density, volume, length of section, respectively. Variable $r_i^{in}$ and $r_i^{out}$ are the inflow and outflow of the ramp of the $i$th section.

The macro traffic model consists of three equations [8, 9]:

$$
\dot{\rho}_i = \frac{1}{L_i} (q_{i-1} - q_i + r_i^{in} - r_i^{out}); \quad (1)
$$

$$
\dot{v}_i = \frac{1}{\tau} (v_i(\rho_i) - v_j) + \frac{v_j}{L_i} (v_{i-1} - v_i) - \frac{\gamma}{\alpha} \frac{\rho_{i+1} - \rho_i}{\rho_i + \lambda}; \quad (2)
$$

$$
q_i = \alpha \rho_i v_i + (1 - \alpha) \rho_{i+1} v_{i+1}. \quad (3)
$$

The drivers within each section accelerate or decelerate according to the conditions of where they placed in and the adjacent sections. In (2), $\tau$ is the delay time, $\gamma$ and $\lambda$ are constants, and the function $v_i(\rho_i)$ is defined as

$$
v_i(\rho) = v_f \left[1 - \left(\frac{\rho}{\rho_{cr}}\right)^m\right], \quad (4)
$$

where $l$ and $m$ are constants, and the variable $\rho_{cr}$ and $v_f$ are the jam density and the maximum speed. In (3), the variable $\alpha$ is a momentum factor.

From (2), the equation added control item is

$$
\dot{v}_i = \frac{1}{\tau} (v_i(\rho_i) - v_j) + \frac{v_j}{L_i} (v_{i-1} - v_i) - \frac{\gamma}{\alpha} \frac{\rho_{i+1} - \rho_i}{\rho_i + \lambda} + u_i; \quad (5)
$$

Defining

$$
f_j = \frac{1}{\tau} (v_i(\rho_i) - v_j) + \frac{v_j}{L_i} (v_{i-1} - v_i) - \frac{\gamma}{\alpha} \frac{\rho_{i+1} - \rho_i}{\rho_i + \lambda}, \quad (6)
$$

then (5) can be transformed as follows

$$
\dot{v}_i = f_j + u_i. \quad (7)
$$

According to (3), it is easy to get

$$
\dot{\rho}_i = \frac{\alpha}{L_i} \rho_{i+1} v_{i+1} - \frac{2\alpha - 1}{L_i} \rho_i v_i + \frac{1}{L_i} \left[1 - (1 - \alpha) \rho_{i+1} v_{i+1} + r_i^{in} - r_i^{out}\right]. \quad (8)
$$

For the sake of simplicity,

$$
\dot{\rho}_i = g_i + e_i, \quad (9)
$$

where

$$
g_i = \frac{\alpha}{L_i} \rho_{i+1} v_{i+1} - \frac{2\alpha - 1}{L_i} \rho_i v_i, \quad (10)
$$

and

$$
e_i = \frac{1}{L_i} \left[1 - (1 - \alpha) \rho_{i+1} v_{i+1} + r_i^{in} - r_i^{out}\right]. \quad (11)
$$

Equations (7) and (9) are the station variation of the $i$th section under the control. The variable $e_i$ in (9) can be considered as disturbance item, and assuming it is bounded, such that

$$
|e_i| \leq \bar{e}_i, \quad (12)
$$

where $\bar{e}_i$ is upper bound of the absolute value of $e_i$.

3. Variable Structure Control
The variable structure control theory is an important part of control theory [10]. In variable structure control, the approaching law is usually used to solving control item in many practice applications. For a nonlinear system

\[ \dot{x} = f(x, u, t), \]  

where \( x \) is state vector, \( u \) is control item. Suppose \( s \) is the switching variable of \( x \), the approaching law has many types, for example

\[ \dot{s} = -\varepsilon \text{sgn} \, s - ks, \varepsilon > 0, k > 0. \]  

It is easy to see that, if \( s \) satisfies (14), \( \dot{s} < 0 \), thus \( s \to 0 \). Because symbols function \( \text{sgn}(s) \) is not continuous, in many situations \( \text{sgn}(s) \) substituted by function

\[
[s \text{gn} s]_{cd} = \begin{cases} 
\text{sgn} s, & |s| \geq \frac{\pi}{2\omega} \\
\sin \omega s, & |s| < \frac{\pi}{2\omega}
\end{cases}
\]  

where \( \omega \) is a constant. The derivative of \([s \text{gn} s]_{cd}\) is

\[
[s \text{gn} s]'_{cd} = \begin{cases} 
\text{sgn} s, & |s| \geq \frac{\pi}{2\omega} \\
\omega \sin \omega s, & |s| < \frac{\pi}{2\omega}
\end{cases}
\]  

4. Solving Control Item

In this paper, the control strategy is using speed controller to stabilize the density within certain range. Firstly, suppose \( s_i \) is switching function of density

\[ s_i = \rho_i^* - \rho_i, \]  

where \( \rho_i^* \) is a constant, and the first step Lyapunov function is as follows

\[ V_i = \frac{1}{2} s_i^2. \]  

Then,

\[ \dot{V}_i = s_i \dot{s}_i = s_i \dot{\rho}_i = s_i \left( \frac{\alpha}{L_i} \rho_{i+1} - \frac{2\alpha - 1}{L_i} \rho_i + e_i \right). \]  

If

\[ v_i^* = \frac{1}{2(\alpha - 1)\rho_i} \left( \alpha \rho_{i+1} - e_i - s_i \right), \]  

Substituting (16) to (15), we can get

\[ \dot{V}_i = -ks_i^2 + e_i s_i - \sigma [s \text{gn}(s_i)]_{cd} s_i. \]  

According to(16), if \( |s_i| \geq \frac{\pi}{2\omega} \), then \([s \text{gn}(s_i)]_{cd} s_i = s_i \), from (12), we get \( e_i s_i \leq \sigma [s \text{gn}(s_i)]_{cd} s_i \), so \( \dot{V}_i \leq -ks_i^2 \). Then, if (16) is satisfied, \( s_i \) is decreasing until \( |s_i| < \frac{\pi}{2\omega} \), and then \( \rho_i - \frac{\pi}{2\omega} < \rho_i < \rho_i^* - \frac{\pi}{2\omega} \).

In (16), \( v_i^* \) can be regarded as target speed. For the speed approaching to the target speed, let

\[ V_i = V_i^* + \frac{1}{2} s_i^2 = \frac{1}{2} s_i^2 + \frac{1}{2} s_i^2, \]  

where

\[ s_i = v_i - v_i^*. \]  

Then

\[ \dot{V}_i = \ddot{V}_i + s_i (\dot{v}_i - \dot{v}_i^*). \]  

Where \( \dot{v}_i \) can be decided by (7), and from (20), \( \dot{v}_i \) decided as follows
\[
\dot{v}_i^* = \frac{\alpha \dot{\rho}_{i-1} v_{i-1} + \alpha \rho_{i-1} \dot{v}_{i-1} + k L_i \dot{\rho}_i + \bar{\sigma}_L \text{sgn}(s_i) \dot{v}_{i-1} + (2\alpha-1)\rho_i}{(2\alpha-1)\rho_i} - \frac{\left[\alpha \rho_{i-1} v_{i-1} + k L_i s_i + \bar{\sigma}_L \text{sgn}(s_i) \rho_i \right]}{(2\alpha-1)\rho_i^2} \cdot \tag{25}
\]

Form (7) and (9), substituting \( \dot{\rho}_i, \dot{\rho}_{i-1}, \dot{v}_{i-1} \) into (25), we get

\[
\dot{v}_i^* = \frac{\alpha \rho_{i-1} v_{i-1} + \rho_{i-1} \dot{v}_{i-1} + h_i}{(2\alpha-1)\rho_i}, \tag{26}
\]

where

\[
h_i = \left[\alpha \rho_{i-1} v_{i-1} + k L_i \rho_i + \bar{\sigma}_L \text{sgn}(s_i) \rho_i \right] - \left[\alpha \rho_{i-1} v_{i-1} + k L_i s_i + \bar{\sigma}_L \text{sgn}(s_i) \rho_i \right] \tag{27}
\]

If

\[
\dot{s}_i = \dot{v}_i - \dot{v}_i^* = -k s_i, \tag{28}
\]

then \( s_i \to 0 \). So, in order to get control item, substituting (26) into (28) and considering (7), we have

\[
f_i + u_i = \frac{\alpha \rho_{i-1} v_{i-1} + \rho_{i-1} \dot{v}_{i-1} + h_i - k s_i}{(2\alpha-1)\rho_i}, \tag{29}
\]

so

\[
u_i = -f_i + \frac{\alpha \rho_{i-1} v_{i-1} + \rho_{i-1} \dot{v}_{i-1} + h_i - k (v_i - v_i^*)}{(2\alpha-1)\rho_i}. \tag{30}
\]

Where \( f_i, h_i \) and \( v_i^* \) are decided by (6), (27) and (20), respectively.

The bound condition is treated as follows: assuming \( \rho_0, v_0 \) are constants, the \( L \)th and \( (L+1) \)th section have the same density and speed \((\rho_{L+1}, v_{L+1}) = (\rho_L, v_L)\). According to (7),

\[
\dot{v}_0 = f_0 + u_0 = 0, \tag{31}
\]

so

\[
u_i = -f_i + h_i - k (v_i - v_i^*). \tag{32}
\]

Equations (30) and (32) are the control items.

**5. Simulation**

In order to prove proposed control strategy, simulation was carried out with Matlab. The parameters are \( L=3, L_i=0.5, \tau=0.008, \lambda=5, \gamma=35, \rho_0=80, \nu_1=110, i=1.7, m=1.8, \alpha=0.99, k=10, \omega=1, r_i^{\text{in}}=r_i^{\text{out}}=0 \) (\( i=1,2,3 \)), \( \bar{\sigma}_1 = 176 \). The specified density was \( \rho^*_i = 30 \), and the initial conditions are \((\rho_i, v_i) = (30, 50) \) (\( i=1, 2, 3 \)). The controller added to the system until \( t=1 \). Fig 1, 2, 3 are shown the variation of the density and speed of each road section. The speed and density fluctuate of the period \( 0 < t < 1 \), finally within a certain range under the control. It proves the effectiveness of the control method.

![Fig. 2. Density and speed curves of the first segment](image1)

![Fig. 3. Density and speed curves of the second segment](image2)
6. Conclusion

This paper discussed a control method of continuous macro traffic model based on variable structure control theory. The speed controller of the road section was designed to stabilize the density of traffic flow. Under the control, variations of density and speed were reduced, and the traffic congestion can be avoided.

7. Acknowledgment

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8. References