Hexagonally Constrained Minimax Design of Variable Fractional-Delay Digital Filters

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Abstract
This paper proposes a new method for designing finite-impulse-response (FIR) variable fractional-delay (VFD) digital filters in the minimax error sense using the well-known linear programming (LP) approach. To convert the original non-linear minimax design into a linear one, we linearize the non-linear constraints as linear constraints through constraining the complex-valued errors of variable frequency response (VFR) within a hexagon in the complex plane. This results in some constraints that are linear functions of the VFD filter coefficients. Thus, the minimax design can be carried out by using the LP technique.

1. Introduction
Variable digital filters are a new type of digital filters whose magnitude responses [1]-[8] or and variable fractional-delay (VFD) responses [9], [10] are instantly changeable. Without redesigning a new digital filter, the users can instantly tune the frequency response of a variable digital filter on-line. VFD filters have found various signal processing applications such as sampling rate conversion [9], discrete-time signal interpolations [10], image interpolation, and fractional-order differentiator design. The simplest VFD filter is the so-called Lagrange-type VFD filter [10], which has the maximally flat frequency characteristic at frequency $\omega = 0$, but its passband width is very narrow. To speed up the Lagrange-type VFD filtering, symmetric structures for both even-order and odd-order Lagrange-type VFD filters have been derived and theoretically proved [11], [12], [13]. To achieve wide-band VFD filtering, various methods have been developed [13]-[24]. Most of the methods are weighted-least-squares (WLS) designs that minimize the total error energy of variable frequency response (VFR).

This paper focuses on the minimax design of FIR VFD digital filters that minimize the maximum absolute error of VFR. Since the VFR error is complex-valued, which includes both real-part and imaginary-part, the minimax design is inherently non-linear. To simplify the non-linear minimax design as a simple (linear) one such that the resulting linear problem can be easily solved by applying the linear programming (LP) techniques, the first step is to convert the non-linear constraints into linear constraints. By doing so, the non-linear minimax design can be linearized as an LP problem and then it can be solved easily. We will show that both the real-part error and imaginary-part error of the VFR error are mutually independent, so the two errors can be separately minimized, i.e., the non-linear minimax design can be decoupled into a pair of separate designs. This decoupling approach is very simple and has very low design burden, but may cause a large deviation from the true minimax design. In the worst case, the largest deviation may be about -3 dB theoretically. The reason for this is that the complex-valued VFR errors are only constrained within a square in the complex plane and if both the real-part error and imaginary-part error reach the maximum, then the absolute VFR error will be $\sqrt{2}$ the real-part error (or imaginary-part error). To improve the design accuracy, this paper proposes a more accurate technique for linearizing the original non-linear constraints. The new minimax design methodology is to constrain the VFR errors within a hexagon. Since a hexagon is closer to a circle (true minimax design) than a square, the largest deviation can be reduced, which leads to a more accurate VFD filter. We will use an example to show that the hexagonally constrained VFD filter is more accurate than the one desgined using the decoupling LP design.

2. Hexagonally Constrained Minimax Design
The ideal VFR of an even-order FIR VFD filter is...
where $\omega \in [0, \alpha \pi]$ is the normalized angular frequency, the parameter $\alpha$ specifies the passband edge frequency $\alpha \pi$, $0 < \alpha < 1$, and $p$ is the VFD parameter, $p \in [-0.5, 0.5]$.

Let the transfer function of an even-order VFD filter be

$$H(z, p) = \sum_{n=-N}^{N} h_n(p) z^{-n}$$

(2)

i.e., we use (2) to approximate (1), where the variable coefficients $h_n(p)$ are the $M$th-degree polynomials in $p$ as

$$h_n(p) = \sum_{m=0}^{M} a(n, m) p^m.$$  

(3)

Substituting (3) into (2) yields

$$H(z, p) = \sum_{n=-N}^{N} \left[ \sum_{m=0}^{M} a(n, m) z^{-n} \right] p^m = \sum_{m=0}^{M} \left[ \sum_{n=-N}^{N} a(n, m) z^{-n} \right] p^m = \sum_{m=0}^{M} F_m(z) p^{2m} + \sum_{m=1}^{M} G_m(z) p^{2m-1}$$

(4)

where $F_m(z)$ and $G_m(z)$ are called sub-filters in the Farrow structure [9],

$$F_m(z) = \sum_{n=-N}^{N} a(n, 2m) z^{-n}, \quad G_m(z) = \sum_{n=-N}^{N} a(n, 2m - 1) z^{-n}$$

(5)

and $\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$ are floor function and ceiling function, respectively. Also,

$$a(-n, 2m) = a(n, 2m)$$

$$a(-n, 2m - 1) = -a(n, 2m - 1).$$

and for $m = 0$, $a(n, m) = \delta(n)$, thus $F_0(z) = 1$. The frequency responses of $F_m(z)$ and $G_m(z)$ are

$$F_m(\omega) = \sum_{n=0}^{N} b_{em}(n) \cos(n\omega) = c_m^T b_{em}$$

$$G_m(\omega) = (-j) \cdot \sum_{n=1}^{N} b_{om}(n) \sin(n\omega) = (-j) \cdot s_m^T b_{om}$$

(7)

where

$$b_{em}(n) = \begin{cases} a(n, 2m) & n = 0 \\
2a(n, 2m), & n = 1, 2, \ldots, N \end{cases}$$

$$b_{om}(n) = 2a(n, 2m - 1), \quad n = 1, 2, \ldots, N$$

and

$$c_m^T = [1 \ \cos(\omega) \ \cos(2\omega) \ \cdots \ \cos(N\omega)]$$

$$s_m^T = [\sin(\omega) \ \sin(2\omega) \ \cdots \ \sin(N\omega)]$$

$$b_{em} = \begin{bmatrix} b_{em}(0) \\
b_{em}(1) \\
\vdots \\
b_{em}(N) \end{bmatrix}, \quad b_{om} = \begin{bmatrix} b_{om}(1) \\
b_{om}(2) \\
\vdots \\
b_{om}(N) \end{bmatrix}.$$
Therefore, we have
\[
F_m(\omega)p^{2m} = \sum_{n=0}^{N} b_{em}(n) \cos(n\omega) \cdot p^{2m} = f_m^T b_{em} 
\]
\[
G_m(\omega)p^{2m-1} = (-j) \cdot \sum_{n=1}^{N} b_{om}(n) \sin(n\omega) \cdot p^{2m-1} = (-j) \cdot g_m^T b_{om} 
\]
with
\[
f_m^T = e_m^T p^{2m}, \quad g_m^T = s_m^T p^{2m-1}.
\]
Hence, the VFR of \(H(z, p)\) can be written as
\[
H(\omega, p) = 1 + \sum_{m=1}^{M_e} f_m^T b_e - j \sum_{m=1}^{M_o} g_m^T b_o = 1 + f^T b_e - jg^T b_o = \text{Re}(\omega, p) - j\text{Im}(\omega, p) 
\]
where
\[
\text{Re}(\omega, p) = 1 + f^T b_e, \quad \text{Im}(\omega, p) = g^T b_o
\]
\[
f^T = [f_1^T \quad f_2^T \quad \cdots \quad f_{M_e}^T], \quad g^T = [g_1^T \quad g_2^T \quad \cdots \quad g_{M_o}^T]
\]
and
\[
b_e = \begin{bmatrix} b_{e1} \\ b_{e2} \\ \vdots \\ b_{eM_e} \end{bmatrix}, \quad b_o = \begin{bmatrix} b_{o1} \\ b_{o2} \\ \vdots \\ b_{oM_e} \end{bmatrix}.
\]
The VFR error can be computed by using (1) and (8) as
\[
e(\omega, p) = H(\omega, p) - H_1(\omega, p) \\
= e_R(\omega, p) - je_1(\omega, p)
\]
with
\[
e_R(\omega, p) = \text{Re}(\omega, p) - \cos(\omega p) \\
e_1(\omega, p) = \text{Im}(\omega, p) - \sin(\omega p).
\]
To constrain the VFR error \(e(\omega, p)\) such that both \(e_R(\omega, p)\) and \(e_1(\omega, p)\) are inside a hexagon as shown in Fig. 1. That is, the maximum deviation from the true minimax design may be \((1 - \cos(\pi/6))\epsilon \approx 0.13\epsilon\). Consequently, the LP design can be formulated as
\[
\text{minimize} \quad \epsilon \\
\text{subject to} \quad |\text{Re}[\epsilon(\omega, p)e^{j\theta_k}]| \leq \epsilon \cos(\pi/6)
\]
with \(\theta_k = k\pi/3, \quad k = 0, 1, 2\), where \(\text{Re}[c]\) denotes the real-part of complex number \(c\). Since
\[
\text{Re}[\epsilon(\omega, p)e^{j\theta_k}] = e_R(\omega, p) \cos \theta_k + e_1(\omega, p) \sin \theta_k \\
= [1 + f^T b_e - \cos(\omega p)] \cos \theta_k + [g^T b_o - \sin(\omega p)] \sin \theta_k
\]
the constraint in (13) can be re-written as
\[
-\epsilon \cos(\pi/6) + \text{Re}[\epsilon(\omega, p)e^{j\theta_k}] \leq 0 \\
-\epsilon \cos(\pi/6) - \text{Re}[\epsilon(\omega, p)e^{j\theta_k}] \leq 0.
\]
Substituting (14) into (15) yields
\[
A_k x \leq d_k, \quad k = 0, 1, 2, 3
\]
with

\[ A_k = \begin{bmatrix} -\cos(\pi/6) & f_k^T \cos \theta_k & g_k^T \sin \theta_k \\ -\cos(\pi/6) & -f_k^T \cos \theta_k & -g_k^T \sin \theta_k \end{bmatrix} \]

\[ x = \begin{bmatrix} \epsilon \\ b_c \\ b_o \end{bmatrix}, \quad d_k = \begin{bmatrix} [\cos(\omega_p) - 1] \cos \theta_k + \sin(\omega_p) \theta_k \\ -[\cos(\omega_p) - 1] \cos \theta_k - \sin(\omega_p) \theta_k \end{bmatrix}. \]

Combining (16) for \( k = 0, 1, 2 \) together arrives at

\[ A = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix}, \quad d = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}. \]

Hence, the LP design (13) becomes

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq d
\end{align*}
\]

with \( c^T = [1 \ 0 \ \cdots \ 0] \).

3. Conclusion

This paper has formulated a hexagonally constrained LP design approach for designing even-order FIR VFD filters. By constraining the complex-valued VFR errors within a hexagon, we can yield more accurate minimax design than constraining the VFR errors within a square. Due to the limitation of space, we cannot provide more detailed simulation results in this paper.

4. References

Fig. 1. Hexagonally constrained VFR errors.